

Phases of massive scalar field collapse

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We study critical behavior in the collapse of massive spherically symmetric scalar fields. We observe two distinct types of phase transition at the threshold of black hole formation. Type II phase transitions occur when the radial extent (λ) of the initial pulse is less than the Compton wavelength (μ^{-1}) of the scalar field. The critical solution is that found by Choptuik in the collapse of massless scalar fields. Type I phase transitions, where the black hole formation turns on at finite mass, occur when $\lambda\mu \gg 1$. The critical solutions are unstable soliton stars with masses $\lesssim 0.6\mu^{-1}$. Our results in combination with those obtained for the collapse of a Yang-Mills field [M. W. Choptuik, T. Chmaj, and P. Bizon, Phys. Rev. Lett. **77**, 424 (1996)] suggest that unstable, confined solutions to the Einstein-matter equations may be relevant to the critical point of other matter models.

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The discovery of critical point behavior in gravitational collapse has highlighted the role played by non-linear dynamics at the threshold of black hole formation, and has opened up a fascinating area of research in General Relativity.

Choptuik [1] performed the first definitive numerical study of critical behavior in the collapse of spherically-symmetric distributions of massless scalar field. His results indicated that one parameter families of interpolating solutions $S[p]$ generically have a critical value $p = p^*$ such that (i) $S[p < p^*]$ are solutions in which the scalar field disperses to infinity, and (ii) $S[p > p^*]$ are solutions in which the field collapses to form a black hole. In slightly super-critical evolutions, Choptuik found that the black-hole mass has a simple power-law form

$$M_{BH} \simeq K|p - p^*|^\gamma, \quad (1)$$

where the critical exponent is $\gamma \simeq 0.37$ and K is a family dependent constant. For near critical evolutions the field asymptotically approaches a discretely self-similar form, with an echoing period $\Delta \simeq 3.44$ which is the same for all families, before either dispersing to infinity or forming a black hole. Based on these observations, Choptuik conjectured that a unique solution to the Einstein-scalar field equations acts as the intermediate attractor for near critical evolutions. Gundlach has directly constructed this critical solution, and has computed the echoing period and the critical exponent to be $\Delta = 3.4453 \pm 0.0005$ and $\gamma = 0.374 \pm 0.001$ [2,3], thus confirming the numerical estimates. It has also been argued that this picture is stable against the introduction of a small scalar field mass [3–5].

Motivated by the results of Choptuik, critical point behavior has also been studied in other models of gravitational collapse [6–9]. Dynamical self-similarity (either discrete or continuous) in near critical evolutions, and a scaling relation for black-hole mass, as in Eq. (1), are

common features of these models, although the numerical value of the critical exponent γ is model dependent. It is now well established that the power law form for the black-hole mass derives from the existence of a single unstable mode of the critical solution in each case [10,11,2]. Of the examples considered to date, the evolution of the Yang-Mills field is exceptional. Choptuik *et al* [12] have studied this model, finding two distinct types of phase transition depending on the initial field configurations they considered. In what they refer to as *Type I* transitions, the black hole formation turns on at finite mass and the critical solution is the Bartnik-McKinnon solution [13]—a regular, static, but unstable, solution to the spherically symmetric Einstein-Yang-Mills equations. In *Type II* transitions black-hole formation turns on at infinitesimal mass and the critical behavior is qualitatively similar to that found by Choptuik for massless scalar fields, except that the scaling exponent is $\gamma \simeq 0.20$ and the echoing period is $\Delta \simeq 0.74$. An independent confirmation of these results has been provided by Gundlach [14].

Here we report on a detailed study of critical phenomenon in the collapse of spherically symmetric configurations of a massive scalar field. The introduction of a mass μ destroys the scale invariance of the Einstein-scalar field equations. Moreover, the massive scalar-field equations admit soliton-like solutions as discussed by Seidel and Suen [15]. These observations suggest that the qualitative picture of critical point behavior could differ from the massless limit, and might be similar to that found by Choptuik *et al.* [12] in their study of Einstein-Yang-Mills collapse*. Our results show that both Type I and Type II

*In this case, the SU(2) Yang-Mills charge breaks the scale invariance of the field equations.

phase transitions occur in the collapse of massive scalar fields. Furthermore, we advance a simple criterion to determine which type of phase transition will be observed for a given initial data set. If the radial extent λ of the initial pulse is greater than the Compton wavelength μ^{-1} of the scalar field then Type I phase transitions will be observed. Type II transitions develop from initial data with $\lambda\mu \lesssim 1$. This criterion provides an intuitive, physical explanation of the observed phenomenology, and clarifies the role played by intrinsic scales in critical collapse.

We write the general, spherically symmetric line-element in terms of a retarded time u and a radial co-ordinate r , which measures proper area of the 2-spheres, as

$$ds^2 = -g\bar{g}du^2 - 2gdudr + r^2d\Omega^2, \quad (2)$$

where g and \bar{g} are functions of both r and u , and $d\Omega^2$ is the line-element on the unit 2-sphere. We choose to normalize u to be proper time at the origin, thus fixing $g(0, u) = 1$. Imposing regularity of the spacetime at the origin requires $\bar{g}(0, u) = 1$. The evolution of the massive scalar field is governed by the wave equation $\square\phi - \mu^2\phi = 0$. It is convenient to introduce an auxiliary field h [16], related to ϕ by

$$\phi = \bar{h} = \frac{1}{r} \int_0^r dr' h(r', u), \quad (3)$$

in terms of which the wave equation can easily be written in a first order form. Thus, the coupled Einstein-scalar field equations are

$$(\ln g)_{,r} = r^{-1}(h - \bar{h})^2, \quad (4)$$

$$(r\bar{g})_{,r} = g(1 - \mu^2 r^2 \bar{h}^2), \quad (5)$$

$$(r\bar{h})_{,r} = h, \quad (6)$$

$$h_{,u} - \frac{\bar{g}}{2}h_{,r} = \frac{1}{2r}(h - \bar{h})[g(1 - \mu^2 r^2 \bar{h}^2) - \bar{g}] - rg\mu^2 \bar{h}/2. \quad (7)$$

The dynamics is completely encompassed in the last equation, which is the wave equation written in terms of the new fields h and \bar{h} .

The characteristic initial value problem requires only the field ϕ to be supplied on some initial outgoing null cone, which we will take, without loss of generality, to be at $u = 0$. We have considered the evolution of three different initial data sets as shown in Table I.

The numerical algorithm used to integrate these equations is documented [17,18] elsewhere. We have followed the scheme as outlined by Garfinkle [18]. The accuracy of the code has been tested previously, where it was used to study radiative tails of a massless scalar field propagating in asymptotically de Sitter spacetimes [19], and was found to be locally second order accurate.

TABLE I. The three initial data sets considered in our evolutions. Only a single parameter is varied when looking for a critical point. The types of phase transitions which may occur are indicated under Type.

	$\phi(u=0, r)$	Parameters	Type
(i)	$\phi_0 r^2 \exp[-(r-r_0)^2/\sigma^2]$	σ, ϕ_0	I, II
(ii)	$\phi_0 \{1 - \tanh[(r-r_0)/\sigma]\}$	σ, ϕ_0	I, II
(iii)	$\phi_0 r(r+r_0)^{-\sigma}/(1+e^r)$	σ, ϕ_0	I, II

It is reasonable to expect that the picture of critical behavior offered by Choptuik [1] is robust against the introduction of a small scalar-field mass [3–5]. More precisely, the evolution of an initial distribution of scalar field will differ from the massless evolution only if the characteristic length-scale μ^{-1} , set by the scalar field mass, is smaller than the radial extent λ of the region in which the field is non-zero*. This expectation is supported by our numerical integrations. We generally observe Type II phase transitions when $\lambda\mu \ll 1$. Furthermore, we find that the scaling relation for the black hole mass is in agreement with the massless limit, having $\gamma \approx 0.378$.

The new feature, arising due to the presence of the mass μ , is the existence of Type I phase transitions—phase transitions in which black hole formation turns on at finite mass. The critical solutions are soliton stars on the unstable branch of the mass versus radius curve discussed by Seidel and Suen [15]. The mass gap at the threshold of black hole formation lies in the range $0.35 \lesssim \mu M_{BH} \lesssim 0.59$, the upper limit being set by the maximum mass that a soliton star can have.

For the three initial data sets we examined (see Table I), we have found both Type I and Type II behavior, along with evidence that both critical solutions play a role when $\lambda\mu \approx 1$. In contrast to the Einstein-Yang-Mills system [12] our results suggest that the shape of the initial data does not determine the critical point behavior.

Physically the existence of the different regimes can be understood if we recall the two known limits for scalar field collapse. In the massless regime $\lambda\mu \ll 1$ an outward pressure is required for the field to bounce back to infinity, whereas in the adiabatic regime $\lambda\mu \gg 1$ the collapse is pressureless [20]. Hence, by continuity in the space of solutions, it seems likely that there can exist configurations, characterized by $\lambda\mu = C \sim \mathcal{O}(1)$, such that the field neither disperses to infinity nor collapses to a

*For generic initial data it is not possible to define the radial extent, however, for the initial data sets (i) and (ii) in Table I, we set $\lambda = 2\sigma$.

singularity.

Figure 1 shows the field ϕ at the origin $r = 0$ during the critical phase of a Type I evolution for the Gaussian initial data in Table I with $r_0 = 5.0$, $\sigma = 1.2$ and $\phi_0 = 6.4746$. The solution behaves like a soliton star with an effective mass $\sim 0.52\mu^{-1}$ confined within a radius $\sim 4\mu^{-1}$ for an amount of time $u \sim 100$. The angular frequency of its fundamental mode is $\omega_0 \approx 1.8\mu$, the next mode is also apparent at $3\omega_0$. Superimposed on these oscillations is an amplitude modulation with period $\sim 15.7\mu^{-1}$ which is reflected in the sideband structure of the Fourier amplitude.

Sub-critical evolutions generally settle down to a scalar field configuration dominated by a single oscillatory mode with angular frequency $\omega \approx 1.05$ as shown in Fig 2 for $\mu = 1.0$. Further exploration indicates that the fundamental oscillations have a period given approximately by $2\pi\mu^{-1}$. Unfortunately, the characteristic evolution scheme makes it difficult to follow the evolutions beyond $u \sim 400$. Nevertheless, the trend suggests that the amplitude of the scalar field is decreasing slowly—indeed this is precisely the regime in which the stationary phase approximation is valid [20].

When the critical solution corresponds to the marginally stable soliton star with effective mass $\sim 0.6\mu^{-1}$, we have found evidence of further phenomenology. In particular, the solutions may closely approach the solitonic configuration, begin to disperse, but recollapse to form black holes. This behavior merits further investigation, however our numerical scheme is not well suited for this purpose.

As evidence of the observed mass gap, we present in Fig. 3 the spectrum of black hole masses near to criticality for the initial data set (iii) of Table I. The black-hole mass at threshold is $M_{BH} \approx 0.51\mu^{-1}$. This mass spectrum is most interesting for what it does not do, rather than what it does. We determine that a solution contains a black hole if either of the metric functions g or \bar{g} exceeds some pre-specified tolerance G_{\max} anywhere on a slice of constant u . Suppose this occurs at $u = u_{BH}[\phi_0]$, where ϕ_0 is the parameter being varied in the initial data. The mass of the black hole is then $M_{BH} = \frac{1}{2}r_{BH}$ where r_{BH} is the location of the *global* minimum of $f(r; \phi_0) \equiv \bar{g}(r, u_{BH}[\phi_0])/g(r, u_{BH}[\phi_0])$ on the slice $u = u_{BH}$. Notice that M_{BH} need not depend continuously on ϕ_0 if $f(r; \phi_0)$ has more than one local minimum. Indeed, as ϕ_0 is varied in our simulations we sometimes observe discontinuities in the black-hole mass when the tolerance G_{\max} is not large enough; this is shown in the inset of Fig. 3. A careful inspection, varying both the numerical resolution and the tolerance, suggests that the discontinuities are *not* a real effect in the black-hole mass spectrum. (Discontinuities in the mass spectrum are also alluded to in [12]; it would be interesting to check if they arise for similar reasons.) In contrast, the oscillation imposed on the mass spectrum

in Fig. 3 is not an artifact of the numerics but is similar to the fine structure found by Hod and Piran [21] in the Choptuik results.

To better understand the selection effect between Type I and Type II phase transitions we have constructed families of interpolating solutions $S_\lambda[\phi_0]$ for several values of λ . Generally, we find Type I transitions occur when the Bondi mass M_{Bondi} of the initial field profile is greater than $\sim 0.4\mu^{-1}$ and its radial extent is larger than the Compton wavelength of the field, i.e. $\lambda\mu \gtrsim 1$. Figure 4 shows the Bondi mass of the initial data at the critical point $\phi_0 = \phi_0^*$, and the resulting black-hole mass, for the initial data sets (i) and (ii) in Table I.

In conclusion, we find that the presence of a length scale changes the nature of critical phenomena in gravitational collapse of a scalar field. It introduces new phenomenology which is similar to that discussed by Choptuik et al [12]. Moreover, it is tempting to speculate that unstable, confined solutions will act as critical solutions in other matter models. We therefore expect that both Type I and Type II phase transitions should occur in the gravitational collapse of perfect fluids (with equations of state which allow stationary configurations), *and* in the collapse of charged massive scalar fields [22].

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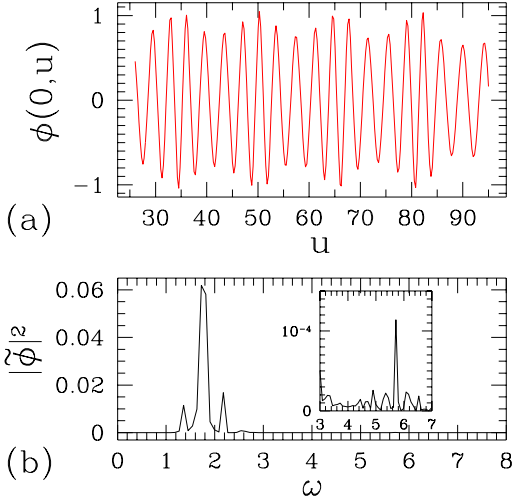


FIG. 1. (a) The time evolution of ϕ as a function of retarded time u at the origin for a near critical evolution with Gaussian initial data (See Table I). It clearly exhibits an underlying periodic solution with a superimposed amplitude modulation. (b) The squared amplitude of the discrete Fourier transform of $\phi(0, u)$. The fundamental oscillation has an angular frequency $\omega_0 \approx 1.8\mu$, and, in agreement with Seidel and Suen [15], the next important feature is at three times this frequency $\omega \sim 5.65\mu$. The sidebands determine the period of the amplitude modulation to be $\approx 15.7\mu^{-1}$.

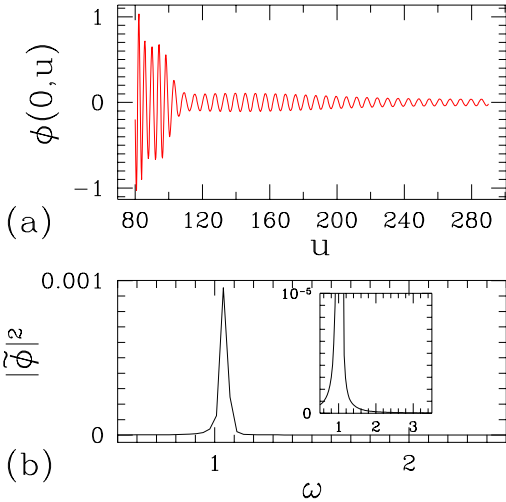


FIG. 2. (a) The time evolution of ϕ as a function of retarded time u at the origin for the same evolution as in Fig 1, but after the nearly critical phase which ends at $u \approx 100$. The solution is essentially periodic, with a low frequency amplitude modulation. (b) The squared amplitude of the discrete Fourier transform of the field for $u \geq 100$. The fundamental oscillation has an angular frequency $\omega_0 \sim 1.05\mu$, which suggests that it is determined by the scalar field mass (which is set to unity). The inset demonstrates that other harmonics are not strongly excited in this solution. For other initial data sets, we find that the amplitude of the oscillations tends to decay slowly, and the solutions are dispersing.

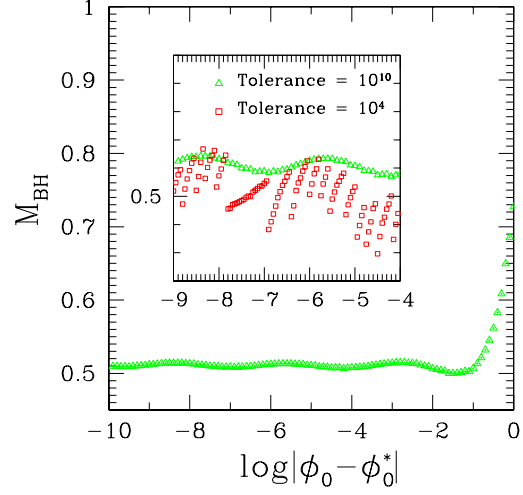


FIG. 3. The black-hole mass M_{BH} as a function of $\log |\phi_0 - \phi_0^*|$ for supercritical evolutions with $\mu = 1.0$. The results displayed are for the initial data set (iii) in Table I with $r_0 = 2.0$ and $\sigma = 10.0$. The critical point was determined to be $\phi_0^* = 3.87245233459$ with an initial radial discretization $\Delta r = 0.05$. The black hole tolerance parameter (see text) was set at 10^{10} . The inset shows results obtained from the same evolution, but with a lower black hole tolerance level of 10^4 . For low values of the tolerance the mass spectrum exhibits *spurious* discontinuities.

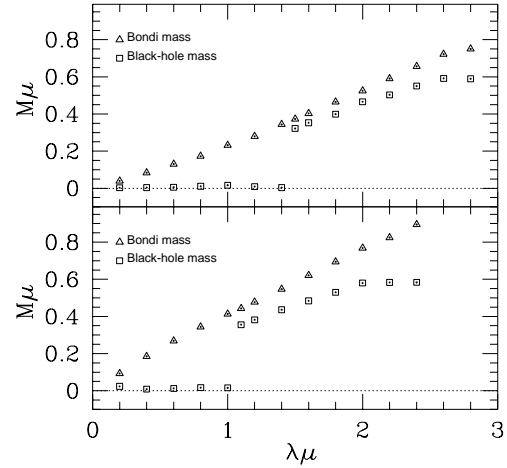


FIG. 4. The Bondi mass of the initial scalar field profile, in practice $m(r, 0) = r(1 - \bar{g}/g)/2$ evaluated at the outer edge of the grid, and the measured black-hole mass at the critical point versus the radial extent λ of the initial profile for the data sets (i) and (ii) in Table I. Type I transitions are evident for $\lambda\mu \gg 1$, and Type II transitions when $\lambda\mu \ll 1$. The interface between Type I and Type II behavior is clearly visible when $M_{\text{Bondi}} \simeq 0.4\mu^{-1}$, and $\lambda\mu \sim 1$.